INTEGRABLE SYSTEMS RELATED TO BRAID GROUPS(Research on Complex Analytic Geometry and Related Topics)

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INTEGRABLE SYSTEMS RELATED TO BRAID GROUPS

(braid群に関連した可積分系)

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INTRODUCTION. This note is a brief review on a recent development in the study of linear representations of the braid groups appearing as the monodromy of certain integrable connections. These connections are defined for any simple Lie algebra and its irreducible representation and appear in a natural way to describe n-point functions in the conformal field theory on the Riemann sphere with gauge symmetry due to Knizhnik and Zamolodchikov [12]. We focus the role of solutions of the *Yang-Baxter equation for the face model* to express the monodromy properties of these n-point functions. We will show that the Markov trace, which plays an important role to construct invariants of links due to Jones [10] and several other authors [1][19][22], appear as "weighted" characters of these monodromy representations. The reader may refer to [15][16] and [21] for a complete exposition on these subjects.

1. INFINITESIMAL PURE BRAID RELATIONS.

We start from a finite dimensional complex simple Lie algebra $\mathfrak{g}$ and its irreducible representation $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$. Let $\{I_\mu\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form. We consider the matrices $\Omega_{\alpha\beta} \in \mathrm{End}(V^{\otimes n})$, $1 \le \alpha < \beta \le n$, defined by

$$\Omega_{\alpha\beta} = \sum_{\mu} 1\otimes \dots \otimes 1\otimes \overset{\alpha}{\rho(I_{\mu})}\otimes 1\otimes \dots \otimes 1\otimes \overset{\beta}{\rho(I_{\mu})}\otimes 1\otimes \dots \otimes 1 \tag{1.1}$$

By using the fact that the Casimir element lies in the center of the universal enveloping algebra $U(g)$ we have the relations:

$$[\,\Omega_{\alpha\beta},\, \Omega_{\alpha\gamma}+\Omega_{\beta\gamma}] = [\,\Omega_{\alpha\beta}+\Omega_{\alpha\gamma},\, \Omega_{\beta\gamma}] = 0 \quad \text{for} \quad \alpha<\beta<\gamma \tag{1.2}$$

$$[\,\Omega_{\alpha\beta},\, \Omega_{\gamma\delta}] = 0 \quad \text{for distinct} \quad \alpha,\ \beta,\ \gamma,\ \delta\ .$$

The above relations can be considered to be a special case of the classical Yang-Baxter equation and have the following significance. Let us consider the 1-form

$$\omega = \sum_{\alpha<\beta} \lambda\, \Omega_{\alpha\beta}\, d\log(z_{\alpha}-z_{\beta})\ . \tag{1.3}$$

with a complex parameter λ over

$$X_n = \{(z_1,\ \dots,\ z_n) \in \mathbb{C}^n\ ;\quad z_{\alpha}\neq z_{\beta} \quad \text{if} \quad \alpha\neq\beta\ \} \tag{1.4}$$

The relations (1.2) show that the connection ω is integrable. The fundamental group of X_n is called the pure braid group with n strings and the quadratic relations (1.2) may be consider to be an infinitesimal version of the defining relations of the pure braid group. This idea to express the relations for the fundamental group by the integrability condition goes back to Poincaré and Cartan. Following the work of Chen [4] and Sullivan [20] we can establish the precise group theoretical meaning of the relations (1.2) ([13]).

The *braid group* B_n is by definition the fundamental group of the quotient $X_n/\mathfrak{S}_n$, where the symmetric group acts as the permutation of the coordinates. Now as the monodromy of the connection ω we obtain a one parameter family of linear representations

$$(1.5)\qquad \varphi : B_n \longrightarrow \mathrm{End}(V^{\otimes n})$$

2. QUANTIZED UNIVERSAL ENVELOPING ALGEBRA AND R-MATRIX.

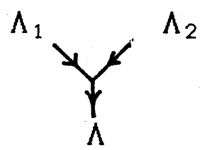
We present a second method to obtain linear representations of B_n Let $U_\hbar(\mathfrak{g})$ be the *quantized universal enveloping algebra* of $\mathfrak{g}$ in the sense of Drinfel'd [5] and Jimbo [6]. We put $q = e^{\hbar/2}$. Let $\rho_i : U_\hbar(\mathfrak{g}) \longrightarrow \mathrm{End}(V_i)$, $i=1,2$, be the irreducible representation with the highest weight Λ_i. The tensor product $V_1 \otimes V_2$ has a structure of $U_\hbar(\mathfrak{g})$-module by means of the comultiplication of $U_\hbar(\mathfrak{g})$. Under the assumption that any irreducible component in $V_1 \otimes V_2$ has multiplicity one, Reshetikhin [19] obtained the following R-matrix.

$$(2.1)\qquad R^{\Lambda_1\Lambda_2} = \sum_\Lambda (-1)^{\varepsilon(\Lambda)} q^{\{c(\Lambda)-c(\Lambda_1)-c(\Lambda_2)\}/2} P_\Lambda^{\Lambda_1\Lambda_2}$$

The meaning of the notations is as follows. First, we define the q-*Clebsch-Gordan coefficient* (see Fig.1)

$$C = C_\Lambda^{\Lambda_1\Lambda_2}(q) \quad : \quad V_1 \otimes V_2 \longrightarrow V_\Lambda$$

Fig.1



for any irreducible module V_Λ with the highest weight Λ contained in $V_1 \otimes V_2$. The row vectors of C consist of the weight vectors of V_Λ and we normalize C as $C\,{}^tC = I$. We define the projector P_Λ

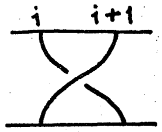
by ${}^{t}C.C$. We put $c(\Lambda) = \langle\Lambda, \Lambda+2\delta\rangle$ where δ is the half sum of the positive roots of $\mathfrak{g}$ and $\varepsilon(\Lambda)$ is the parity of V_Λ in $V_1 \otimes V_2$.

In the case $\mathfrak{g}$ is non-exceptional and Λ_i , $i=1,2$, corresponds to the vector representation the above R-matrix is extracted from trigonometric solutions of the Yang-Baxter equation

$$R_{12}(u)R_{23}(u+v)R_{12}(v) = R_{23}(v)R_{12}(u+v)R_{23}(u) \tag{2.3}$$

due to Jimbo [7] by tending the spectral parameter u to the infinity.

Let us suppose $\Lambda_1 = \Lambda_2$. We denote by σ_i , $1 \le i \le n-1$, the standard generators of B_n (see [2]). Then the corresponsence



$$\sigma_i \longrightarrow R_{i,i+1} = 1\otimes \ldots \otimes 1 \otimes \overset{i,i+1}{R} \otimes 1 \otimes \ldots \otimes 1$$

where $R = R^{\Lambda_1\Lambda_2}$ gives a linear representation of B_n denoted by $\pi : B_n \longrightarrow \mathrm{End}(V^{\otimes n})$. This representation commutes with the diagonal action of $U_\hbar(\mathfrak{g})$ and if we consider the classical limit $\hbar \to 0$ the above construction gives the situation due to Brauer and Weyl.

<u>3. FUSION PATH AND NORMALIZED SOLUTIONS.</u> In this section, we start from the vector representation $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ and we suppose that $q = e^{\hbar/2}$ is not a root of unity. We denote by Π the highest weight of the vector representation. We suppose that $q = e^{\hbar/2}$ is not a root of unity. The n-fold tensor product $V^{\otimes n}$ has a decomposition $\oplus\, (M_\Lambda \otimes V_\Lambda)$ as a $U_\hbar(\mathfrak{g})$-module where M_Λ stands for the multiplicity of the representation V_Λ corresponding to the highest weight Λ . We have a basis of M_Λ described in the following way.

Let $\mathcal{P}(\Lambda)$ denote the set of the sequence $(\Lambda_0, \ldots, \Lambda_n)$ of dominant integral weights of $\mathfrak{g}$ satisfying the following:

(3.1) (i) $\Lambda_0 = 0$, $\Lambda_n = \Lambda$

(ii) $V_{\Lambda_i} \otimes V$ contains $V_{\Lambda_{i+1}}$ as a $\mathfrak{g}$-module.

An element p of $\mathcal{P}(\Lambda)$ is called a *fusion path*, which corresponds to some shortest path in the decomposition diagram of $V^{\otimes n}$ as a $\mathfrak{g}$-module. We assocoate to p the following composition of q-Clebsch-Gordan coefficients (see Fig.2 and 5)

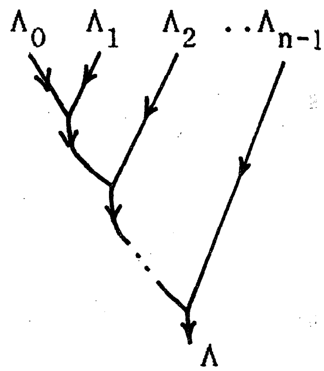


Fig.2

$$(3.2)\quad C^{\Lambda_0 \pi}_{\Lambda_1}(q) \ :\ V^{\otimes n} \longrightarrow V_{\Lambda_1} \otimes V^{\otimes(n-2)}$$

$$C^{\Lambda_1 \pi}_{\Lambda_2}(q) \ :\ V_{\Lambda_2} \otimes V^{\otimes(n-2)} \longrightarrow V_{\Lambda_3} \otimes V^{\otimes(n-3)}$$

$$\cdots\cdots$$

$$C^{\Lambda_{n-1} \pi}_{\Lambda}(q) \ :\ V_{\Lambda_{n-1}} \otimes V \longrightarrow V_{\Lambda}$$

which defines a projector $e_p(q) \ :\ V^{\otimes n} \longrightarrow V_\Lambda$. These $e_p(q)$ form a basis of M_Λ and the action of the braid group is expressed by using W defined by

$$(3.3)\quad W\begin{pmatrix} \Lambda_{i-1} & , \Lambda'_i \\ \Lambda_i & , \Lambda_{i+1} \end{pmatrix} = e_p(q) R_{i,i+1}\, {}^t e_{p'}(q)$$

(see Fig.3). We have

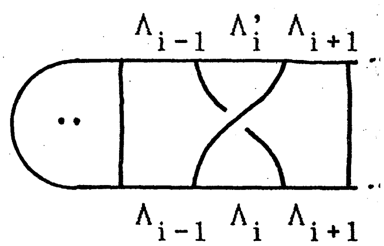


Fig.3

$$(3.4)\quad \sigma_i \,.\, e_p(q) = \sum_{p'} W\begin{pmatrix} \Lambda_{i-1}, & \Lambda'_i \\ \Lambda_i & , \Lambda_{i+1} \end{pmatrix} e_{p'}(q)$$

where the RHS is the sum with respect to $\hbar' = (\Lambda'_0, \ldots, \Lambda'_n) \in \mathcal{P}(\Lambda)$ satisfying $\Lambda_j \neq \Lambda'_j$ if $j \neq i$. The above coefficients W satisfies the Yang-Baxter equation for the face model

$$(3.5) \qquad \sum_g W\begin{pmatrix} f, & g \\ e, & d \end{pmatrix} W\begin{pmatrix} b, & c \\ g, & d \end{pmatrix} W\begin{pmatrix} a, & b \\ f, & g \end{pmatrix}$$

$$= \sum_g W\begin{pmatrix} a, & b \\ g, & c \end{pmatrix} W\begin{pmatrix} a, & g \\ f, & e \end{pmatrix} W\begin{pmatrix} g, & c \\ e, & d \end{pmatrix}$$

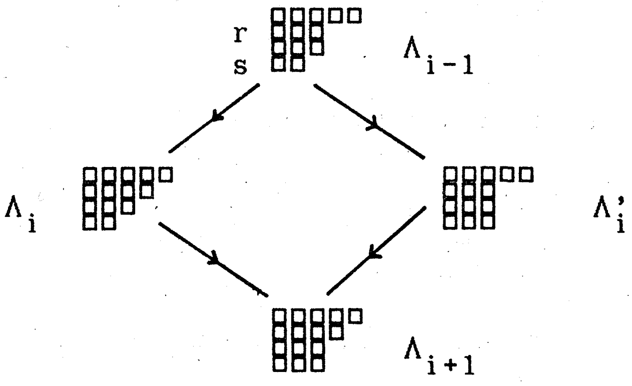
and they are extracted from the Boltzmann weights for the IRF model due to Jimbo, Miwa and Okado [8] by taking the critical limit and tending the spectral parameter to the infinity.

example. Let us consider the case $\mathfrak{g} = \mathfrak{sl}(N,\mathbb{C})$. We suppose that Λ_{i-1} corresponds to the Young diagram of type $(d_1, \ldots, d_m)$, $d_1 \geq \ldots \geq d_m \geq 0$ and Λ_i and Λ'_i are obtained by adjoining one node to the r-th and s-th column $(r \neq s)$ respectively (see Fig.4). We put $d = (d_r - r) - (d_s - s)$. In this case W is given by

$$(3.6) \qquad W\begin{pmatrix} \Lambda_{i-1} & , \Lambda'_i \\ \Lambda_i & , \Lambda_{i+1} \end{pmatrix} = \sqrt{[d-1][d+1]/[d]^2}$$

where $[k]$ stands for $(q^k - q^{-k})/(q - q^{-1})$.

Fig.4



The above description of the action of the braid group by means of the face language can be used effectively to describe the monodromy of ω defined in (1.3) with respect to the *normalized solutions associated with the fusion paths* in the following sense. Let $w_1, \ldots, w_{n-1}$ be the blowing up coordinates of X_n such that $w_k=0$ corresponds to $z_1= \ldots = z_{k+1}$. The residue of ω along $w_k=0$ is given by $\sum_{1\leq\alpha<\beta\leq k+1} \Omega_{\alpha\beta}$. These elements are diagonalized simultaneouly with respect to the basis $e_{\hbar} = \lim_{q\to 1} e_{\hbar}(q)$ with the eigenvalues

$$(3.7) \qquad \mu_k = \frac{1}{2}\lambda \{c(\Lambda_{k+1}) - (k+1)c(\Pi)\}, \quad 1\leq k\leq n-1 ,$$

Let us suppose that $q = \exp(\pi i\lambda)$ is not a root of unity. Then the total differential equation $d\Phi = \omega\Phi$ has solutions associated with the fusion paths given by

$$(3.8) \qquad \phi_{\hbar}(z)= w_1^{\mu_1} w_2^{\mu_2} \ldots w_{n-1}^{\mu_{n-1}} \{ e_{\hbar} + (\text{higher order terms}) \}$$

We can show (see [16]) that after a certain normalization $\phi_{\hbar}^{\sim}(z) = \alpha_{\hbar}(\lambda)\phi_{\hbar}(z)$ the monodoromy of the braid group is expressed as

$$(3.9) \qquad \sigma_i^* \phi_{\hbar}^{\sim}(z) = \sum_{\hbar'} W\begin{pmatrix} \Lambda_{i-1}, & \Lambda_i' \\ \Lambda_i & , \Lambda_{i+1} \end{pmatrix} \phi_{\hbar'}^{\sim}(z)$$

To show this we used the following expansion

$$(3.10) \qquad (R_{12}R_{23} \ldots R_{k-1,k})^k = 1 + \hbar \sum_{1\leq\alpha<\beta\leq k} \Omega_{\alpha\beta} + O(\hbar^2)$$

together with a description of the Riemann-Hilbert correspondence for the pure braid group obtained by investigating the group theoretical meaning of the infinitesimal pure braid relations (see [14]).

4. FUSION METHODS. The following priciple to compute the monodromy by localizing the situation to the case of four variables was discovered by Tsuchiya and Kanie [21]. We start from $\mathfrak{g} = \mathfrak{sl}(N,\mathbb{C})$ and its vector representation. We consider the fusion paths connecting Λ_{i-1} and Λ_{i+1} (see Fig.4). Such a fusion path $\hbar$ defines a $\mathfrak{g}$-homomorphism $e_\hbar : V_{i-1}\otimes V\otimes V \longrightarrow V_{i+1}$ where V_j denotes the irreducible representation with the highest weight Λ_j. We have

$$(4.1) \qquad \Omega_{i-1,i}\, e_\hbar = \frac{1}{2}\{c(\Lambda_i) - c(\Pi) - c(\Lambda_{i-1})\}\, e_\hbar$$

$$(\Omega_{i-1,i}+\Omega_{i-1,i+1}+\Omega_{i,i+1})\, e_\hbar = \frac{1}{2}\{c(\Lambda_{i+1}) - 2c(\Pi) - c(\Lambda_{i-1})\}\, e_\hbar$$

Let us denote by $\Delta\, e_\hbar$ the RHS of the second equation. Let $\omega\hat{}$ be the connection defined by

$$(4.2) \qquad \omega\hat{} = \textstyle\sum_{i-1\le\alpha<\beta\le i+1} \lambda\, \Omega_{\alpha\beta}\, d\log(z_\alpha - z_\beta)$$

We put $z_{i-1} = 0$. The total differential equation $d\Phi = \omega\hat{}\Phi$ can be written in the form

$$(4.3) \qquad \frac{d}{d\xi}\Psi_0(\xi) = \lambda\,\{\,\Omega_{i-1,i}/\,\xi + \Omega_{i,i+1}/(\xi-1)\,\}\,\Psi_0(\xi)$$

Here we put $\Phi(z_i, z_{i+1}) = z_{i+1}^{\Delta}\,\Psi_0(\xi)$, $\xi = z_i/z_{i+1}$.
In our case this is essentially the Gauss hypergeometric differential equation and by means of the classical methods we can compute the

matrix relating the solution Ψ_0 normalized at 0 and the solution Ψ_∞ normalized at the infinity. This method enables us to express the normalizing factor $\alpha_{\hbar}(\lambda)$ appiearing in the previous section by means of the Γ functions in the following way. In the situation of Fig.4 we define $\gamma_i(\hbar)$ to be

$$\gamma_i(\hbar) = \Gamma(\lambda d)/\sqrt{\Gamma(\lambda(d-1))\,\Gamma(\lambda(d+1))}$$

If there is no such $\Lambda_i'\neq\Lambda_i$, we put $\gamma_i(\hbar)=1$. Then the gamma factor $\alpha_{\hbar}(\lambda)$ is given by the product $\gamma_1(\hbar)\ \ldots\ \gamma_n(\hbar)$.

This principle can be also applied to higher representations of $\mathfrak{g}$. We have a formula analogous to (3.9) where W is computed from the R-matrix associated with higher representaitions. Let us note that the linear representations of the braid groups defined by this R-matrix were used by Akutsu and Wadati [1] and Murakami [17] to construct invariants of links.

5. ALGEBRAS FACTORING THROUGH THE MONODROMY. We suppose that $\mathfrak{g}$ is a non-exceptional simple Lie algebra and $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ its vector representation. In the case of $\mathfrak{g}$ is of type A the monodromy representation φ is equivalent to the higher order Temperley-Lieb representation and it factors through the Iwahori's Hecke algebra. In the other cases φ factors through a specialization of the algebra with two parameter $\mathscr{C}_n(\alpha,\beta)$ discovered by Birman-Wenzl [3] and Murakami [18]. These algebras are denoted by $\mathscr{C}_n(\mathfrak{g},q)$ and may be considered to be a q-analogue of Brauer's centralizer algebras.

The following Markov trace was related to invariants of links by Jones [10] and Turaev [22].

$$(5.1) \qquad \tau(x) = \chi^{-n}\, \mathrm{Tr}((q^{-\delta}|V)^{\otimes n}.\varphi(x)) \qquad \text{for} \quad x \in B_n$$

where δ is the half sum of the positive coroots and χ is $\mathrm{Trace}(q^{-\delta}|V)$. The above τ gives a functional on $\mathscr{C}_\infty(\mathfrak{g},q)$.

We will see in the next section that the case q is a root of unity is important from the viewpoint of the conformal field theory. In this case the algebra $\mathscr{C}_n(\mathfrak{g},q)$ is not semi-simple but the above Markov trace gives us a method to construct its semi-simple quotient. Let us suppose that $q = \exp(\pi i/(\ell+g))$ where ℓ is a positive integer called a level and g is the corresponding dual Coxeter number (see [11]). We consider

$$(5.2) \qquad J_n = \{\ x \in \mathscr{C}_n(\mathfrak{g},q)\ ;\ \tau(xy) = 0 \quad \text{for any} \quad y \in \mathscr{C}_n(\mathfrak{g},q)\ \}$$

Then it turns out that the quotient algebra $\overline{\mathscr{C}}_n = \mathscr{C}_n(\mathfrak{g},q)/J_n$ is semi-simple. The irreducible representations of this algebra are described in the following way. Let $\mathscr{P}_\ell(\Lambda)$ be the subset of $\mathscr{P}(\Lambda)$ consisting of $\not{h} = (\Lambda_0, \ldots, \Lambda_n)$ such that $\langle\Lambda_i,\theta\rangle \le \ell$, for any i, where θ denotes the highest root and the Cartan-Killing form is normalized as $\langle\theta,\theta\rangle = 2$. For $\not{h}$, $\not{h}' \in \mathscr{P}_\ell(\Lambda)$, we put

$$(5.3) \qquad w_{\not{h}'\not{h}} = \lim_{q\to\xi}{}^{\cdot}\ W\begin{pmatrix} \Lambda_{i-1} & , & \Lambda'_i \\ \Lambda_i & , & \Lambda_{i+1} \end{pmatrix}$$

where $\xi = \exp(\pi i/(\ell+g))$. It turns out that the above limit is a non-zero finite number. Then the representations of B_n given by

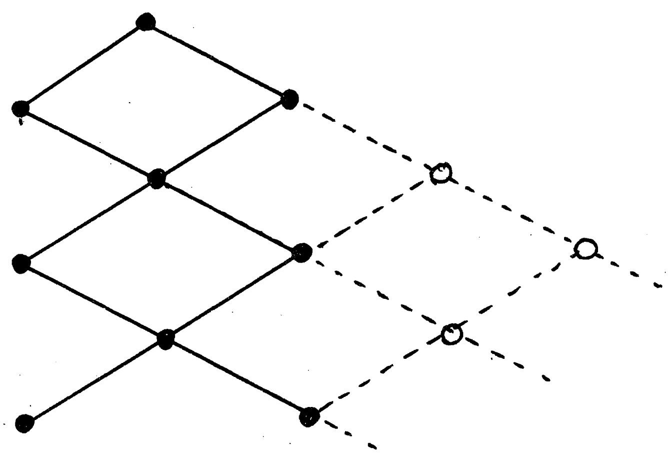
$$(5.4) \qquad \sigma_i \cdot e_{\hbar} = \sum_{\hbar'} w_{\hbar'\hbar}\, e_{\hbar'}$$

give all irreducible representations of $\overline{\mathscr{C}}_n$. The above construction corresponds to the *restricted model* in the terminology of the solvable lattice models (see [8]).

Moreover, we can show that the Markov trace τ defines a positive definite bilinear form on the algebra $\overline{\mathscr{C}}_n$.

example. We illustrate some examles of the decomposition of the algebra $\overline{\mathscr{C}}_n$ in the following figure.

Fig.5 $\mathfrak{g}=sl(2,\mathbb{C})$, $\ell=2$



6. MONODROMY OF n-POINT FUNCTIONS. We discuss how the framework described in the previous sections can be applied to illustrate the monodromy properties of the n-point functions in the conformal field theory on the Riemann sphere with gauge symmetry of affine Lie algebras. We refer to [12] and [21] for the operator formalism in this theory, which we shall review briefly.

Let $\mathfrak{g}^{\wedge}$ denote the affine Lie algebra associated with $\mathfrak{g}$, which is defined to be the canonical central extension of the loop algebra $\mathfrak{g}\otimes\mathbb{C}[t, t^{-1}]$. Starting from a finite dimensional irreducible $\mathfrak{g}$-module V whose highest weight Λ satisfies $\langle\Lambda,\theta\rangle\leq\ell$ it is known by Kac [11] that we can associate an irreducible $\mathfrak{g}^{\wedge}$-module generated by

$$X_1(-n_1)\, X_2(-n_2)\, \ldots\, X_k(-n_k).\, v \quad , \quad X_i \in \mathfrak{g} \; , \; n_i>0 \; , \quad 1\leq i\leq k \; ,$$

for $v \in V_\Lambda$ on which the central element $\hat{c}$ of $\mathfrak{g}^{\wedge}$ acts as $\ell\times id$. Here $X(n)$ stands for $X\otimes t^n$. This is called the *integrable highest weight module of level* ℓ with the highest weight Λ and is denoted by $\mathcal{H}_\Lambda$. The *Sugawara form*

$$(6.1) \qquad L_n = \frac{1}{2(\ell+g)} \sum_\mu \sum_{k\in\mathbb{Z}} :I_\mu(-k)\, I_\mu(n+k):$$

satisfies the relation of the *Virasoro Lie algebra*

$$(6.2) \qquad [\, L_m \, , \, L_n \,] = (m-n)\, L_{m+n} + \frac{m^3-m}{12}\, \delta_{m+n,0}\, c$$

with the central charge $c = \ell \dim \mathfrak{g}/(\ell+g)$. A *primary field* $\Phi(u,z)$ is an operator on $\oplus_{\langle\Lambda,\theta\rangle\leq\ell}\, \mathcal{H}_\Lambda$ depending linearly on $u \in V_\Pi$ with some fixed Π, depending holomorphically on $u \in \mathbb{C}-\{0\}$ and satisfying the following conditions.

$$(6.3) \qquad [\, L_m \, , \, \Phi(u,z) \,] = z^m\{z(\partial/\partial z) + (m+1)\Delta_\Pi\}\, \Phi(u,z)$$

with $\Delta_\Pi = c(\Pi)/(\ell+g)$,

(6.4) $\quad [\ X(m),\ \Phi(u,z)\ ] = z^{m}\ \Phi(Xu,z) \quad$ for $\ X \in \mathfrak{g}$.

For simplicity we suppose that V_{π} is the vector representation. Assocoated with an ℓ-contraint fusion path $\not{h}=(\Lambda_0, \ \ldots, \ \Lambda_n) \in \mathcal{P}_{\ell}(\Lambda)$ Tsuchiya and Kanie constructed a vertex operator Φ_i for each i which is a primary field sending $\mathcal{H}_{\Lambda_i}$ to $\mathcal{H}_{\Lambda_{i+1}}$.

It was shown by Knizhnik and Zamolodchikov [12] that the n-point function

(6.5) $\quad \phi_{\not{h}}(z) = \langle u|\Phi_n\Phi_{n-1}\ \ldots\ \Phi_1|vac\rangle \quad , \ u \in V_{\Lambda}^{\dagger}$

is a solution of the total differential $d\Phi = \omega\Phi$ where ω is defined in (1.3) with the parameter $\lambda = 1/(\ell+g)$. It turns out that the above n-point function is the normalized sulution associated with the fusion path $\not{h}$ in the sense of Section 3 . Now the monodromy of the above n-pont functions is described in the following way. We have a non-zero constant $\alpha_{\not{h}}$ such that the monodromy is expressed by using $w_{\not{h}',\not{h}}$ defined in (5.3) as

(6.6) $\quad \sigma_i^{*}\ \alpha_{\not{h}}\phi_{\not{h}}(z) = \sum_{\not{h}'} w_{\not{h}',\not{h}}\ \alpha_{\not{h}'}\phi_{\not{h}'}(z)$

in the case $\mathfrak{g}$ is non-exceptional. Consequently the monodromy of n-point functions factors through the semi-simple algebra $\overline{\mathcal{C}}_n$ defined in the previous section carrying a positive Markov trace. This algebra coincides with the Jones algebra of index $4\cos^2(\frac{1}{\ell+2})$ in the case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ (see [9],[21] and [23]).

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